To find the common point of these lines equate the two expressions, separate reals and imaginaries and solve for x and y. Result:

$$x = y = 2R \cos \frac{\theta_4 + \theta_3 - \theta_2 - \theta_1}{2}.$$

When the exponential values are inserted for the sines and cosines, the expression for the common point of W_4 and W_3 becomes

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4}).$$

The symmetry of this expression shows that the point is the common point of the four W lines.

$$\{1, 2, 3, 4\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_5}),$$

$$\{1, 2, 3, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_4}),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{2, 3, 4, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_1}).$$

These 5 points lie on the circle whose center is the point, $\frac{R}{2}\sum_{p=1}^{p=5} e^{i\theta_p}$, and whose radius is $R/2$. Where ϕ is variable, the circle

$$[1, 2, 3, 4, 5] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5}) + \frac{R}{2}e^{i\phi},$$

$$[1, 2, 3, 4, 6] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$[2, 3, 4, 5, 6] \text{ is } \frac{R}{2}(e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi}.$$

These 6 circles have the common point,

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}).$$

In the next step the 7 points, of which $\{1, 2, 3, 4, 5, 6\}$ is one, lie on the circle whose center is $\frac{R}{2}\sum_{p=1}^{p=7} e^{i\theta_p}$ and whose radius is $R/2$. Then follows a group of 8 such circles having the common point, $\frac{R}{2}\sum_{p=1}^{p=8} e^{i\theta_p}$, etc., etc.

In general, with an even number of initial points this method yields a point. The vector to this point from the origin is half the sum of the vectors to the initial points. With an odd number of initial points we have a circle whose radius is half that of the original circle and the vector to whose center is half the sum of the vectors to the initial points. The "W" points on this circle are in perspective relation to the initial points.

CALCULUS.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

I. Solution by H. S. Uhler, Yale University.

Let a denote the apothem of the right-section of the prism. The required area will be expressed as a function of a, n, and R. This area may be analyzed as $4n$ triangles on the surface of

the sphere. Each triangle is bounded by arcs of two great circles and one small circle. The plane of one great circle contains the axis of the prism and the apothem to the chosen lateral face. The other great circle is determined by the same axis and a lateral edge of the selected face. The small circle is the intersection of the same lateral face and the spherical surface.

An element of surface of the sphere may be written $R^2 \sin\phi\, d\phi d\theta$, where ϕ is the polar angle reckoned from the axis of the prism and θ is the azimuth with respect to the plane containing the apothem. From the geometry of the problem it is readily seen that the total area required is expressed by

$$A \equiv 4nR^2 \int_0^{\pi/n} d\theta \int_0^{\Phi} \sin\phi\, d\phi, \text{ where } \sin\Phi = \frac{a}{R}\sec\theta;$$

whence

$$A = 4nR^2 \int^{\pi/n} \left(1 - \frac{1}{R}\sqrt{R^2 - a^2\sec^2\theta}\right) d\theta$$

or

$$A = 4\pi R^2 - 4nR \int_0^{\pi/n} (\sqrt{R^2 - a^2\sec^2\theta}) d\theta.$$

The indefinite integral may be evaluated as follows. Let $y = \sin\theta$, then

$$\int (\sqrt{R^2 - a^2\sec^2\theta})d\theta = \int \frac{(\sqrt{(R^2 - a^2) - R^2y^2})dy}{1 - y^2}$$

$$= \frac{1}{2}\int \frac{(\sqrt{(R^2 - a^2) - R^2y^2})dy}{1 + y} + \frac{1}{2}\int \frac{(\sqrt{(R^2 - a^2) - R^2y^2})dy}{1 - y}.$$

The integral

$$U \equiv \int \frac{(\sqrt{\alpha^2 - y^2})dy}{1 + y} = \int \frac{(\sqrt{\alpha^2 - 1 + 2z - z^2})dz}{z} \quad \text{if} \quad z = 1 + y.$$

Formula 187 of B. O. Peirce's *A Short Table of Integrals* gives

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \int \frac{dz}{\sqrt{\alpha^2 - 1 + 2z - z^2}} - (1 - \alpha^2)\int \frac{dz}{z\sqrt{\alpha^2 - 1 + 2z - z^2}}.$$

The first and second of these integrals can be reduced by formulae 161 and 183 respectively, so that

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \sin^{-1}\left(\frac{z - 1}{\alpha}\right) - (\sqrt{1 - \alpha^2})\sin^{-1}\left(\frac{z + \alpha^2 - 1}{\alpha z}\right).$$

In like manner, the substitution $u = 1 - y$ gives

$$\int \frac{(\sqrt{\alpha^2 - y^2})dy}{1 - y} = -\sqrt{\alpha^2 - 1 + 2u - u^2} - \sin^{-1}\left(\frac{u - 1}{\alpha}\right) + (\sqrt{1 - \alpha^2})\sin^{-1}\left(\frac{u + \alpha^2 - 1}{\alpha u}\right).$$

Returning to θ and noting that $\alpha = (1/R)\sqrt{R^2 - a^2}$ we find

$$\int (\sqrt{R^2 - a^2\sec^2\theta})d\theta = R\sin^{-1}\left(\frac{R\sin\theta}{\sqrt{R^2 - a^2}}\right)$$

$$+\frac{a}{2}\sin^{-1}\left[\frac{R^2(1 - \sin\theta) - a^2}{R\sqrt{R^2 - a^2}(1 - \sin\theta)}\right] - \frac{a}{2}\sin^{-1}\left[\frac{R^2(1 + \sin\theta) - a^2}{R\sqrt{R^2 - a^2}(1 + \sin\theta)}\right].$$

Making use of the relation $\sin^{-1}\beta - \sin^{-1}\gamma = \sin^{-1}(\beta\sqrt{1 - \gamma^2} - \gamma\sqrt{1 - \beta^2})$ the following form (which I have verified by differentiation) is obtained, namely

$$\int (\sqrt{R^2 - a^2\sec^2\theta})d\theta = R\sin^{-1}\left(\frac{R\sin\theta}{\sqrt{R^2 - a^2}}\right) - \frac{a}{2}\sin^{-1}\left[\frac{2a(\tan\theta)(\sqrt{R^2 - a^2\sec^2\theta})}{R^2 - a^2}\right].$$

Finally $\infty > n \geqq 3$

$$A = 4\pi R^2 - 4nR^2\sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{\sqrt{R^2 - a^2}}\right) + 2anR\sin^{-1}\left[\frac{2a\left(\tan\frac{\pi}{n}\right)\left(\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}\right)}{R^2 - a^2}\right].$$

REMARKS. In the preceding analysis it has been tacitly assumed that we are dealing with the general case where the small circles intersect in real points thus forming a scalloped polar zone. Keeping n and R constant and increasing a, from a sufficiently small value, a time will come when the small circles will be externally tangent to one another. This will happen when the lateral edges of the prism are tangent to the sphere, that is, when the cross-section of the prism becomes inscribed in a *great* circle. Then $a = R \cos(\pi/n)$. This equation also follows at once from the second term of the final formula for A since the greatest value attainable by the sine of an angle is unity. In other words, the equation

$$\frac{R \sin(\pi/n)}{\sqrt{R^2 - a^2}} = 1$$

leads to $a = R \cos(\pi/n)$. Again, this relation affords a means of checking the formula for A. Substituting $R \cos(\pi/n)$ for a in function A and taking π for the value of $\sin^{-1} 0$, we find

$$A = 4\pi R^2 - 2\pi n R^2 + 2\pi n R^2 \cos(\pi/n) = 4\pi R^2(1 - n \sin^2(\pi/2n)).$$

Precisely the same result is obtained from the theorem that the area of a zone of one base is equal to the product of its altitude ($R - R\cos(\pi/n)$) by the circumference of a great circle. When a starts from zero and increases to

$$\frac{R \cos(\pi/n)}{\sqrt{1 + \sin^2(\pi/n)}}$$

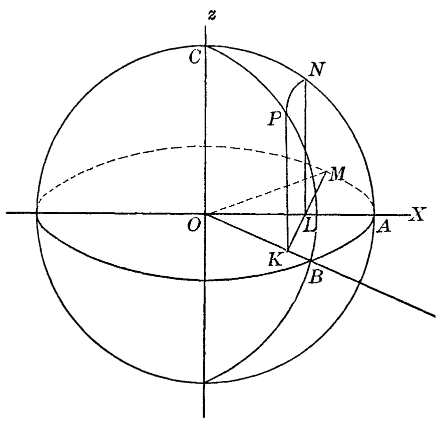
the expression in brackets in the last term of the formula for A increases from 0 to 1. Then as a continues to increase up to $R \cos(\pi/n)$ the expression between the brackets decreases from 1 to 0. Therefore the angle increases continuously from 0 to π. When a exceeds $R \cos(\pi/n)$ the small circles do not intersect, the formula for A does not apply, and the problem degenerates to the calculation of areas of zones of one base. It may also be remarked that the formula for A can be rationalized by making use of the relation $2 \sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

A very convincing check on the correctness of the formula for A is afforded by making n infinite while keeping a and R constant. We should then find $A = 4\pi R^2 - 4\pi R(R^2 - a^2)^{\frac{1}{2}}$ for the sum of the areas of the two zones cut out by a coaxial cylinder of radius a. By expanding in infinite series the functions involving n and then substituting $n = \infty$ we obtain at once the limits $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}}$ and $+4\pi a^2 R(R^2 - a^2)^{-\frac{1}{2}}$ for the second and third terms of the expression for A, respectively. But $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}} + 4\pi a^2 R(R^2 - a^2)^{-\frac{1}{2}} = -4\pi R(R^2 - a^2)^{\frac{1}{2}}$, as required.

II. SOLUTION BY THE PROPOSER.

Let O, the center of the given sphere, be the origin of Cartesian coördinates; the axis OC, of the prism, the z-axis; the line OX through L, the mid-point of a side of the regular polygon formed by the intersection of the prism and a plane through O perpendicular to the axis of the prism, the x-axis; and OY, $\perp OA$ and OC, the y-axis. Call OL, the apothem of this polygon, b.

The equation of the sphere is then $x^2 + y^2 + z^2 = R^2$ and the equation of the line OB is $y = x \tan(\pi/n) = mx$, say. Then the surface of the sphere included by the prism is $4n$ times the area of the triangle PCN. The required surface is, therefore,



$$S = 4n\int_0^b\int_0^{mx}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = 4n\int_0^b\int_0^{mx}\frac{Rdxdy}{\sqrt{R^2-x^2-y^2}}$$

$$= 4nR\int_0^b \sin^{-1}\frac{y}{\sqrt{R^2-x^2}}\Big]_0^{mx}dx = 4nR\int_0^b\sin^{-1}\left(\frac{mx}{\sqrt{R^2-x^2}}\right)dx,$$

$$= 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} - mR\int\frac{xdx}{(R^2-x^2)\sqrt{R^2-(1+m^2)x^2}}\right]_0^b,$$

by integrating by parts.

To integrate the last expression, let $x = R\sin\theta/\sqrt{1+m^2}$. The integral then becomes

$$-mR\int\frac{\sin\theta d\theta}{1+m^2-\sin^2\theta} = mR\int\frac{-\sin\theta d\theta}{m^2+\cos^2\theta} = R\tan^{-1}\left(\frac{\cos\theta}{m}\right) = R\tan^{-1}\left[\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right].$$

Hence,

$$S = 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} + R\tan^{-1}\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right]_0^b$$

$$= 4nR\left\{b\sin^{-1}\left(\frac{mb}{\sqrt{R^2-b^2}}\right) - R\left[\tan^{-1}\left(\frac{1}{m}\right) - \tan^{-1}\frac{\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\frac{mb}{\sqrt{R^2-b^2}} - R\sin^{-1}\left[\frac{m}{1+m^2}\left(\frac{R-\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right)\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\tan\left(\frac{\pi}{n}\cdot\frac{b}{\sqrt{R^2-b^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-b^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-b^2}}\right]\right\}.$$

When $n = 4$,

$$S = 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - 2R\sin^{-1}\left(\frac{1}{2}\cdot\frac{R-\sqrt{R^2-2b^2}}{\sqrt{R^2-b^2}}\right)\right],$$

$$= 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - R\sin^{-1}\left(\frac{b^2}{R^2-b^2}\right)\right],$$

a result agreeing with that in Osborne's *An Elementary Treatise on the Differential and Integral Calculus*, p. 270, example 5. When $n \doteq \infty$, the prism $\doteq$ a cylinder and

$$S \doteq 4R\left[\frac{\pi b^2}{\sqrt{R^2-b^2}} - \pi R\left(\frac{R-\sqrt{R^2-b^2}}{\sqrt{R^2-b^2}}\right)\right] = 4\pi R\left[\frac{b^2-R^2}{\sqrt{R^2-b^2}}+R\right] = 4\pi R(R-\sqrt{R^2-b^2}),$$

that is, twice the circumference of a great circle of the sphere times the altitude of the zone, or the surface of two zones of altitude $(R-\sqrt{R^2-b^2})$.

Also solved by J. W. Clawson.

381. Proposed by ELBERT H. CLARK, Purdue University.

Of all points having the same latitude and a constant difference α in their longitudes, to find the latitude of the two so situated that the distance between them, measured along their common parallel of latitude, shall exceed the distance between them measured on their great circle by the greatest possible amount.

Solution by W. C. Eells, U. S. Naval Academy.

Let L be latitude of the points, 2λ (in radians) their difference in longitude, $2d$ and $2D$ the small circle latitude and great circle distances between them, and $y = d - D$. Let the earth be a sphere of radius unity. Since the radius of the latitude circle is $\cos L$,

$$d = \lambda\cos L. \qquad (1)$$